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N/D Equations with a Finite Strip (*).

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Summary. — A rather complete treatment is given here of N/D equations where the D function has only a finite cut. The original work of Chew on such equations is reviewed and expanded to prove the existence of solutions whenever $\delta_l(s_1) < \pi$ and to reduce the integral equation for N in these cases to a combined Wiener-Hopf-Fredholm type. An explicit formula for the Wiener-Hopf resolvent kernel is obtained which involves a single integral over products of hypergeometric functions. CDD ambiguities are investigated and maximal analyticity of the second degree, or analyticity in angular momentum, is demonstrated as a tool for removing all ambiguity from the solution.

1. Introduction.

We discuss here in some detail the mathematical properties of N/D equations for elastic, partial-wave amplitudes where the D-function has only a finite cut from $s = s_0$ to $s = s_1$ (s is the total center-of-mass energy squared and s_0 is threshold). The N-function carries the right-hand cut of the amplitude above s_1, as well as the left-hand cut. Such strip equations form the basis for at least two current schemes for dynamical calculations in S-matrix theory, that of CHEW (1) and CHEW and JONES (2), and one of BALAZS (3).

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(1) G. F. CHEW: *Phys. Rev.*, **129**, 2363 (1963).

(2) G. F. CHEW and C. E. JONES: *Phys. Rev.*, **135**, B 208 (1964). For numerical results, see D. C. TEPLITZ and V. L. TEPLITZ: *Phys. Rev.*, **137**, B 142 (1965).

(3) L. A. P. BALAZS: *Phys. Rev.*, **134**, B 1315 (1964).

In the strip form of N/D equations, the resulting integral equation for $N_l(s)$ has only a finite-range integration s_0 to s_1 and inelastic effects (such as Regge behaviour [1,2]) can easily be incorporated above s_1.

One of our major concerns here shall be the uniqueness of the solutions to the strip equations. It is well-known that ordinary N/D problems are afflicted with CDD ambiguities [4], which may be characterized by the addition to the D-function of poles whose positions and residues are essentially arbitrary. We wish to understand the role of CDD-type arbitrariness in the strip equations in which elastic unitarity is not assumed throughout the inelastic region. Although a modification of the standard N/D equations to incorporate inelastic effects has been made by several authors [5], still the strip equations appear to provide a new and useful viewpoint for studying the problem. Such a study is useful because of the current application of strip equations in making calculations [1-3].

Our goal here is to indicate how the N/D equations with a finite strip determine one unique solution for the partial-wave amplitude when one insists that amplitudes at lower angular momenta be connected to those at high angular momenta by analytic continuation. This is sometimes called the principle of maximal analyticity of the second degree. In arriving at our goal, we give a rather complete mathematical discussion of the strip equations.

We briefly outline the contents of what follows. First, we give a derivation of the strip equations, in which no assumptions need be assumed about the asymptotic behaviour in energy of the partial-wave amplitudes. In the spirit of the strip concept, this information is contained in the input to the problem.

We continue by studying the existence of solutions to the equations for large l. The full machinery of the Wiener-Hopf-Fredholm theory, originally discussed by CHEW [6], is developed. CHEW proved the existence of solutions for $\delta_l(s_1) < \pi/2$ (where s_1 is the strip edge); we expand his results to demonstrate the existence of solutions whenever $\delta_l(s_1) < \pi$.

The uniqueness of solutions at high l is established by requiring that the amplitude possess no poles (bound states) for large l. The continuation of these solutions to lower values of l is then discussed. We observe, in detail, how zeros of the D-function may move onto the physical sheet as l is decreased but we conclude that the D-function cannot develop poles at small values of l.

The relationhsip of the general results derived here to the models given in ref. [1-3] is mentioned briefly at several points in the text.

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(4) L. CASTILLEJO, R. DALITZ and F. DYSON: *Phys. Rev.*, **101**, 453 (1956).

(5) M. FROISSART: *Nuovo Cimento*, **22**, 193 (1961); G. FRYE and R. L. WARNOCK: *Phys. Rev.*, **130**, 478 (1963). (This paper contains an exhaustive study of the mathematics of partial-wave dispersion relations and N/D equations including a discussion of CDD ambiguities in inelastic problems.)

(6) G. F. CHEW: *Phys. Rev.*, **130**, 1264 (1963).

2. – Derivation of strip equations.

The N/D equations with a finite strip were first written down by CHEW [1] although they are identical in form to equations formulated by URETSKY [7] for an infinite interval. We present here a derivation of the finite-strip equations in which only properties of the amplitude in a finite region including the strip (s_0, s_1) are needed. The high-energy properties and contours of the amplitude are contained in an input function $B_l^r(s)$, which is assumed to be given.

We consider an elastic partial-wave amplitude $B_l(s)$, which is defined in terms of the phase shift $\delta_l(s)$ as follows

$$B_l(s) = \frac{\sin \delta_l(s) \exp[i\delta_l(s)]}{\varrho_l(s)}, \tag{2.1}$$

where

$$\varrho_l(s) = \frac{\nu^{l+\frac{1}{2}}}{\sqrt{\nu+1}} \qquad = (s/4) - 1 . \tag{2.2}$$

For simplicity the four particles are taken to be spinless and of unit mass. The phase-space factor $\varrho_l(s)$ has been chosen so that $B_l(s)$ is real in the gap $0 < s < s_0$ for real values of l [8].

The basic problem of interest is a discussion of the solutions to the equation

$$B_l(s) = B_l^r(s) + \frac{1}{\pi}\int\limits_{s_0}^{s_1} ds' \frac{\operatorname{Im} B_l(s')}{s' - s} \tag{2.3}$$

with the assumption that $B_l^r(s)$ is a given function and that $B_l(s)$ is subjected to the elastic (δ_l real) unitarity restraint (2.1) in the strip (s_0, s_1). We assume temporarily that we are operating at values of l such that there are no poles (bound states) of $B_l(s)$ on the physical sheet. From the well-known analyticity properties of $B_l(s)$ in s we see that $B_l^r(s)$ includes both the left-hand cut of $B_l(s)$ and the right-hand cut above $s = s_1$.

To find a solution of eqs. (2.1) and (2.3) we write

$$B_l(s) = \frac{N_l(s)}{D_l(s)}, \tag{2.4}$$

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(7) J. L. URETSKY: *Phys. Rev.*, **123**, 1459 (1961).
(8) A. O. BARUT and D. ZWANZIGER: *Phys. Rev.*, **127**, 974 (1962).

where $D_l(s)$ is cut from s_0 to s_1 and is real outside this region, while $N_l(s)$ carries the remaining cuts of $B_l(s)$ and is real in the region $0 < s < s_1$.

The justification for the break-up of $B_l(s)$ in (2.4) is provided by the Omnès formula [9]. For sufficiently large l such that there are no bound states we may define

$$D_l(s) = \exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\delta_l(s')}{s'-s}\right]. \tag{2.5}$$

We take for large l the phase-shift convention $\delta_l(s_0) = 0$. The $D_l(s)$ defined in (2.5) clearly carries the phase of $B_l(s)$ on the interval (s_0, s_1), is real outside this interval, and if $\delta_l(s_1) < \pi$, it has no poles or zeros. Finally, $D_l(s) \to 1$ as $s \to \infty$. Thus $D_l(s)$ has the dispersion relation

$$D_l(s) = 1 + \frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\operatorname{Im} D_l(s')}{s'-s}. \tag{2.6}$$

Using (2.3) we may write for $N_l(s)$,

$$N_l(s) = B_l(s)\,D_l(s) = B_l^P(s)\,D_l(s) + \frac{D_l(s)}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\operatorname{Im} B_l(s')}{s'-s}. \tag{2.7}$$

By definition, $N_l(s)$ is real in the interval (s_0, s_1), so the second term in (2.7) must cancel the imaginary part of the first term. We recall that the second term vanishes at infinity like $1/s$ which leads to the unambiguous identification

$$\frac{D_l(s)}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\operatorname{Im} B_l(s')}{s'-s} = -\frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{B_l^P(s')\operatorname{Im} D_l(s')}{s'-s}. \tag{2.8}$$

Finally, we may write for $N_l(s)$, incorporating (2.6) and assuming elastic unitarity in the interval (s_0, s_1),

$$N_l(s) = B_l^P(s) + \frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{B_l^P(s') - B_l^P(s)}{s'-s}\,\varrho_l(s')\,N_l(s'). \tag{2.9}$$

This type of integral equation for $N_l(s)$ with $s_1 = \infty$ was first written down by Uretsky [7]. The use of the equation with $s_1 < \infty$ was first discussed by Chew [1]. The derivation given here is advantageous because we are never

[9] R. Omnès: *Nuovo Cimento*, 21, 524 (1961).

required to discuss explicitly the high-energy properties of the amplitude. All such knowledge is carried along by means of the function $B_l^F(s)$.

Equation (2.9) gives a linear integral equation for $N_l(s)$. If it is soluble, then $D_l(s)$ is determined by the equation

$$D_l(s) = 1 - \frac{1}{\pi}\int_{s_0}^{s_1} \frac{\mathrm{d}s'}{s'-s}\,\varrho_l(s')\,N_l(s')\,. \tag{2.10}$$

It is easy to check that the resulting $B_l(s) = N_l(s)/D_l(s)$ satisfies eqs. (2.1) and (2.3) for real l provided that $N_l(s)$ is real for s in the interval (s_0, s_1).

3. – The problem of solving the integral equation for N.

We discuss here the problem of solving eq. (2.9) and the consequent determination of the partial-wave amplitude. The nature of the solutions—their existence and uniqueness—is clearly determined to a large extent by the input function $B_l^F(s)$. Generally $B_l^F(s)$ will have a singularity at $s = s_1$ but will otherwise be analytic in the strip. This fact follows from eq. (2.3) together with the known analyticity properties of $B_l(s)$.

In the Balazs [3] approximation, $B_l^F(s)$ is assumed to be regular at $s = s_1$. In this case eq. (2.9) becomes Fredholm-type giving a unique solution for $N_l(s)$ which is also nonsingular at $s = s_1$ but $D_l(s)$ as computed by (2.10) will be logarithmically singular at $s = s_1$. This has the effect of forcing the amplitude artificially to vanish at $s = s_1$.

In the CHEW-JONES [2] model, $B_l^F(s)$ is logarithmically singular at $s = s_1$ and eq. (2.9) is no longer Fredholm-type. If s_1 is less than the first inelastic threshold, the exact $B_l^F(s)$ is also logarithmically singular at $s = s_1$ as we shall thortly demonstrate. Thus the model of Chew and Jones [2] may be considered so more nearly approximate the exact strip equations. By studying the realistic case where $B_l^F(s)$ is logarithmically singular at $s = s_1$, while harder, we have the advantage of retaining the features of the exact problem and we know that the answers to the questions of existence and uniqueness of solutions we find will be quite general.

As mentioned, for the case which we now consider, eq. (2.9) is no longer Fredholm. CHEW has studied eq. (2.9) and has shown how to calculate the solution for the case $\delta_l(s_1) < \pi/2$ [6]. CHEW selects the physically interesting solution for this case although we shall show that other solutions do exist which are unitary and that have the correct discontinuities. We shall demonstrate in the following Sections that basically two types of ambiguities arise in defining solutions to the strip equations. One is of the well-known CDD type. The other type of ambiguity, which is peculiar to the finite strip, has to do with properties of the equation at the strip boundary $s = s_1$.

Our goal will be to show that both types of ambiguity are removed by the assumption of maximal analyticity of the second degree [10]—namely that we select solutions under the criterion that they be analytic continuations from arbitrarily large values of angular momentum l.

In order to carry out our program we review the work of Chew [6] in determining solutions to (2.9) and in the process generalize his results to show how to calculate the solutions for all $\delta(s_1)$ less than π.

We begin our discussion by considering the dynamical equations for large l. We shall then formulate the means for determining solutions at lower values of l by analytic continuation.

For sufficiently large l we may assume that there are no bound states and, as before, we take the convention $\delta_l(s_0) = 0$ as $l \to \infty$. It is also true that any physically reasonable $B_l^p(s)$ will have the property

$$B_l^p(s) \xrightarrow[l\to\infty]{} 0 . \tag{3.1}$$

Condition (3.1) is equivalent to the physical requirement that the interaction range be finite. With the convention that $\delta_l(s_0) = 0$ for large l, we may conclude from (3.1) that $\delta_l(s_1) \to 0$ as $l \to \infty$. Now we derive the most general solution of eq. (2.9) for $N_l(s)$ for l large.

First, we observe from eq. (2.3) that as $s \to s_1$ from above, the second term has the limiting behavior [9]

$$\frac{1}{\pi}\int_{s_0}^{s_1} ds' \frac{\operatorname{Im} B_l(s')}{s'-s} \xrightarrow[s\to s_1]{} \frac{\operatorname{Im} B_l(s_1)}{\pi} \ln (s-s_1) . \tag{3.2}$$

So in order for unitarity to be preserved at $s = s_1$, it follows that

$$B_l^p(s_1) \xrightarrow[s\to s_1]{} -\frac{\operatorname{Im} B_l(s_1)}{\pi} \ln (s_1-s) .$$

Thus the kernel of our original equation (2.9) behaves like

$$\frac{\ln (s_1-s') - \ln (s_1-s)}{s'-s}$$

as both s and s' approach s_1 and is not square integrable or of the Fredholm type.

([10]) G. F. Chew: Lawrence Radiation Laboratory Report UCRL-10786 (April 1963) (unpublished).

Following CHEW [6] it is advantageous to separate out the troublesome part of the kernel and to center effort on solving the resulting integral equation

(3.3) $$N_l(s) = N_l^0(s) - \frac{\lambda_l}{\pi^2}\int_{s_0}^{s_1} ds'\, k(s, s')\, N_l(s')\,,$$

where

(3.4) $$N_l^0(s) = B_l^P(s) + \int_{s_0}^{s_1} ds'\, \bar{K}_l(s, s')\, N_l(s')$$

and

(3.5) $$k(s, s') = \frac{\ln(s_1 - s') - \ln(s_1 - s)}{s' - s}\,,$$

(3.6) $$K_l(s, s') = \frac{1}{\pi}\,\frac{B_l^P(s) - B_l^P(s')}{s' - s}\,\varrho_l(s) + \frac{\lambda_l}{\pi^2}\, k(s, s')\,,$$

(3.7) $$\lambda_l = \varrho_l(s_1)\,\mathrm{Im}\, B_l^P(s_1) = \sin^2\delta_l(s_1)\,.$$

The first part of the problem consists in finding the solution to (3.3) which is an integral equation of the Wiener-Hopf type. The procedure is to construct a resolvent kernel $O_l(s, s')$ for the eq. (3.3) so that the solution may be written

(3.8) $$N_l(s) = \int_{s_0}^{s_1} ds'\, O_l(s, s')\, N_l^0(s')\,.$$

One may then combine the result (3.8) together with (3.4) to obtain an equation for $N_l^0(s)$;

(3.9) $$N_l^0(s) = B_l^P(s) + \int_{s_0}^{s_1} ds'\, \widetilde{K}_l(s, s')\, N_l^0(s')\,,$$

where

(3.10) $$\widetilde{K}_l(s, s') = \int_{s_0}^{s_1} ds''\, K_l(s, s'')\, O_l(s'', s')\,.$$

Equation (3.9) will turn out to be Fredholm type with its solution giving $N_l^0(s)$; we then determine $N_l(s)$ through (3.8).

In order to solve eq. (3.3) it is useful to introduce the change of variables [6]

$$x = \ln\left(\frac{s_1 - s_0}{s_1 - s}\right) \tag{3.11}$$

which then gives for (3.3)

$$n_l(x) = n_l^0(x) + \frac{\lambda_l}{\pi^2}\int_0^\infty \mathrm{d}x' \frac{x'-x}{\exp[x'-x]-1}\, n_l(x') \tag{3.12}$$

with $n_l(x) = N_l(s(x))$. Equation (3.12) is an integral equation of the Wiener-Hopf type. The next Section is devoted to discussing the most general solution to (3.12).

4. – The Wiener-Hopf equation.

We now apply the Wiener-Hopf method to the solving of eq. (3.12). The Wiener-Hopf technique [11] consists in defining $n_l^+(x)$ and $n_l^-(x)$ as follows [6];

$$\begin{cases} n_l^+(x) = n_l(x)\,, & x > 0\,,\\ \qquad\;\; = 0\,, & x < 0\,,\\ n_l^-(x) = 0\,, & x > 0\,,\\ \qquad\;\; = n_l(x)\,, & x < 0\,. \end{cases} \tag{4.1}$$

We may thus write

$$n_l(x) = n_l^+(x) + n_l^-(x)\,. \tag{4.2}$$

We next adopt for convenience the convention that

$$n_l^0(x) = 0\,, \qquad x < 0\,. \tag{4.3}$$

Taking the Fourier transform of eq. (3.12) gives [6]

$$g_l^+(k)\left[1 - \frac{\sin^2 \delta_l(s_1)}{\sin^2 \pi i k}\right] + g_l^-(k) = g_l^0(k)\,, \tag{4.4}$$

where g_l^+, g_l^-, g_l^0 are the Fourier transforms of n_l^+, n_l^-, n_l^0.

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(11) E. C. Titchmarsh: *Introduction to the Theory of Fourier Integrals* (London, 1948), p. 339.

Generally speaking, these Fourier tranforms are holomorphic in certain upper or lower half-planes of the complex k variable. In particular g^- and g^0 will be holomorphic functions of k in the following half-planes [6]:

$$g_l^-(k) \text{ holomorphic for } \operatorname{Im} k < 1 ,$$

$$g_l^0(k) \text{ holomorphic for } \operatorname{Im} k > 0 .$$

This follows from the asymptotic behavior of $n_l^0(x)$ and $n_l^-(x)$ in x. The function

$$U(l, k) = 1 - \frac{\sin^2 \delta_l(s_1)}{\sin^2 i\pi k} \tag{4.5}$$

is clearly holomorphic in the strip $0 < \operatorname{Im} k < 1$; we shall succeed in solving the equation if a $g_l^+(k)$ can be found which is holomorphic in a region of the k-plane that overlaps the strip $0 < \operatorname{Im} k < 1$.

It is clear that the function $U(l, k)$ has two zeros in the strip $0 < \operatorname{Im} k < 1$ and we now write $U(l, k)$ in a way which displays the two zeros explicitly:

$$U(l, k) = \frac{\varphi_l^+(k)}{\varphi_l^-(k)} (-ik - a_l)(1 + ik - a_l) , \tag{4.6}$$

where $a_l < 1$ and

$$\left\{\begin{aligned} \varphi_l^+(k) &= \frac{\Gamma^2(-ik)}{\Gamma(a_l - ik)\Gamma(1 - ik - a_l)} , \\ \varphi_l^-(k) &= \frac{\Gamma(1 + a_l + ik)\Gamma(2 + ik - a_l)}{\Gamma^2(1 + ik)} . \end{aligned}\right. \tag{4.7}$$

The function $U(l, k)$ is completely symmetric in the quantities a_l and $(1 - a_l)$ so without loss of generality we may assume that $a_l < \frac{1}{2}$. The $\varphi^\pm$ are written in terms of the gamma-functions and it is readily seen (using the fact that $\Gamma(z)$ is a meromorphic function with simple poles at $z = 0, -1, -2, \ldots$) that φ^+ is holomorphic and free from zeros for $\operatorname{Im} k > 0$ and the same for φ^- in the region $\operatorname{Im} k < 1$.

Since we are discussing the problem at large l values, with our conventions on the phase shift we may identify

$$a_l = \frac{\delta_l(s_1)}{\pi} . \tag{4.8}$$

We further record the important facts that

$$(4.9)\qquad \begin{cases} \varphi_l^+(k)\xrightarrow[k\to\infty]{}\dfrac{1}{|k|} & \text{except for}\quad k\to-i\infty\,, \\ \varphi_l^-(k)\xrightarrow[k\to\infty]{}|k| & \text{except for}\quad k\to+i\infty\,. \end{cases}$$

First we investigate the possibility of homogeneous solutions; setting $g_l^0(k)=0$ in (4.4) gives

$$(4.11)\qquad g_l^+(k)\varphi_l^+(k)(-ik-a_l)(1+ik-a_l)=-g_l^-(k)\varphi_l^-(k)\,.$$

The integrability of $n_l(x)$ at $x=0$ means that

$$(4.12)\qquad g_l^-(k)\xrightarrow[k\to\infty]{}\frac{1}{|k|}\quad\text{in the upper half-plane}\,.$$

If the left and right sides of (4.11) agree in a strip then it is clear that they must be holomorphic in the entire k-plane. Using (4.12) and (4.9) it is readily seen that both sides must be equal to some constant C everywhere. Thus

$$(4.13)\qquad g_l^+(k)=\frac{C}{\varphi_l^+(k)(-ik-a_l)(1+ik-a_l)}\,.$$

We see that $g_l^+(k)$ is holomorphic for $\operatorname{Im}k>1-a_l$ and that $g_l^+(k)\to1/|k|$ as $k\to\infty$ in the upper half-plane. Hence, we do, indeed, have a nontrivial solution to the homogeneous equation given by

$$(4.13)\qquad n_l^H(x)=\frac{C}{(2\pi)^{\frac12}}\int\limits_{-\infty+i(1-\varepsilon)}^{\infty+i(1-\varepsilon)}\mathrm{d}k\,\frac{\exp[-ikx]}{\varphi_l^+(k)(-ik-a_l)(1+ik-a_l)}\,.$$

Next we study the solutions of the inhomogeneous equation. We must, therefore, consider the term $g_l^0(k)\varphi_l^-(k)$ in eq. (4.10). The finiteness of the inhomogeneous term $n_l^0(x)$ at $x=0$ in eq. (3.12) implies

$$(4.14)\qquad g_l^0(k)\xrightarrow[k\to\infty]{}\frac{1}{|k|}\quad\text{in the upper half }k\text{-plane}\,.$$

Recalling (4.9) we have

$$(4.15)\qquad g_l^0(k)\varphi_l^-(k)\xrightarrow[k\to\infty]{}\text{constant}$$

for any direction within the strip of holomorphy $1 > \mathrm{Im}\, k > 0$. Now $g_l^0(k)\varphi_l^-(k)$ may be broken up into a sum of two functions, $\eta^+(k)$ and $\eta^-(k)$, having the property that $\eta^+(k)$ is holomorphic for $\mathrm{Im}\, k > 0$ and that $\eta^-(k)$ is holomorphic for $\mathrm{Im}\, k < 1$. To accomplish this we write

$$g_l^0(k)\varphi_l^-(k) = \frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{\mathrm{d}k'}{k'-k}\, g_l^0(k')\varphi_l^-(k') - \frac{1}{2\pi i}\int\limits_{-\infty+i(1-\varepsilon)}^{\infty+i(1-\varepsilon)} \frac{\mathrm{d}k'}{k'-k}\, g_l^0(k')\varphi_l^-(k')\,, \tag{4.16}$$

which is just Cauchy's identity. Although the integrands in (4.16) each approach a constant, it is easy to verify that the integrals still converge. The indentification of $\eta^{\pm}(k)$ is made as follows:

$$\left\{\begin{aligned} \eta_l^+(k) &= \frac{1}{2\pi i}\int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{\mathrm{d}k'}{k'-k}\, g_l^0(k')\varphi_l^-(k')\,, \\ \eta_l^-(k) &= -\frac{1}{2\pi i}\int\limits_{-\infty+i(1-\varepsilon)}^{\infty+i(1-\varepsilon)} \frac{\mathrm{d}k'}{k'-k}\, g_l^0(k')\varphi_l^-(k')\,. \end{aligned}\right. \tag{4.17}$$

It is also true that $\eta_l^{\pm}(k) \underset{k\to\infty}{\leqslant}$ constant in their respective half-planes of holomorphy. In analogy with eq. (4.11), we can now write

$$g_l^+(k)\varphi_l^+(k)(-ik-a_l)(1+ik-a_l) - \eta_l^+(k) = \eta_l^-(k) - g_l^-(k)\varphi_l(k)\,. \tag{4.18}$$

Here, again, we can argue that both sides of (4.18) must be actually equal to a constant, C_2, thus yielding [12]

$$g_l^+(k) = \frac{C_2 + \eta_l^+(k)}{\varphi_l^+(k)(-ik-a_l)(1+ik-a_l)}\,. \tag{4.19}$$

Equation (4.19) represents the most general solution of (3.12). Adding an arbitrary multiple of the homogeneous solution (4.13) merely changes the constant C_2. So we have

$$n_l(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int\limits_{-\infty+i(1-\varepsilon)}^{\infty+i(1-\varepsilon)} \mathrm{d}k\, \frac{\exp[-ikx]\big(C_2+\eta^+(k)\big)}{\varphi_l^+(k)(-ik-a_l)(1+ik-a_l)}\,. \tag{4.20}$$

([12]) CHEW: [in ref. ([6])] did not consider the general solution given by eq. (4.19) but, instead proceeded in a somewhat different manner in which the arbitrary constant C_2 never appears. However, by the time we reach eq. (4.26) we are in agreement with Chew's result.

At this point it appears as though C_2 enters into the solution of the integral equation as an arbitrary constant. However, we shall now see that C_2 is in fact uniquely determined. The integrand in (4.20) has two poles in the strip $0 < \operatorname{Im} k < 1$, located at $k = ia_l$ and $k = i(1-a_l)$. We have assigned the pole label a_l so that $a_l < \frac{1}{2}$. This means that the asymptotic behaviour of the solution (4.20) is

$$n_l(x) \xrightarrow[x\to\infty]{} \text{const}\cdot\exp[(1-a_l)x]\,. \tag{4.21}$$

This behavior results from taking the residue of the pole in (4.20) at $k = i(1-a_l)$. Because of the dispersion relation (2.10) linking N and D, it follows that D also has the asymptotic behavior (4.21). Transforming back to the s-variable, this gives

$$D_l(s) \xrightarrow[s\to s_1]{} \text{const}\cdot\frac{1}{(s_1-s)^{1-a_l}}\,. \tag{4.22}$$

But here we arrive at a contradiction for eq. (2.5) is also a valid representation of D and gives [6]

$$D_l(s) \xrightarrow[s\to s_1]{} \text{const}\cdot\frac{1}{(s_1-s)^{\delta_l(s_1)}}\,. \tag{4.23}$$

But analyticity in l and the convention that $\delta_l(s) \xrightarrow[l\to\infty]{} 0$ enables us to identify $a_l = \delta(s_1)/\pi$, so behavior (4.22) is wrong. The correct behavior (4.23) can be regained from (4.20) if we select the constant C_2 so that the residue of the pole at $k = i(1-a_l)$ vanishes. This is accomplished by requiring

$$C_2 = -\eta_l^+(k - i(1-a_l))\,. \tag{4.24}$$

Then the leading behavior for $n_l(x)$ becomes $\exp[a_l x]$ given by the pole in (4.20) at $k = ia_l$ and the behavior (4.23) for D results. We can now rewrite $g_l^+(k)$ taking into account (4.24). First we note

$$\eta_l^+(k) + C_2 = \frac{1}{2\pi i}\int_{\infty+i\varepsilon}^{-\infty+i\varepsilon} \mathrm{d}k'\, g_l^0(k')\varphi_l^-(k')\left[\frac{1}{k'-k} - \frac{1}{k'-i(1-a_l)}\right] = \frac{[k-i(1-a_l)]}{2\pi i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathrm{d}k'\,\frac{g_l^0(k')\varphi_l^-(k')}{(k'-k)[k'-i(1-a_l)]}\,. \tag{4.25}$$

This gives the following result for $g_l^+(k)$:

$$g_l^+(k) = \frac{-1}{2\pi}\cdot\frac{1}{\varphi_l^+(k)(-ik-a_l)}\int_{+\infty+i\varepsilon}^{-\infty+i\varepsilon} \mathrm{d}k'\,\frac{g_l^0(k')\varphi_l^-(k')}{(k'-k)[k'-i(1-a_l)]}\,. \tag{4.26}$$

The solution, $n_l(x)$, is then given by the formula

$$n_l(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int\limits_{-\infty+i(1-\varepsilon)}^{\infty+i(1-\varepsilon)} \mathrm{d}k \exp[-ikx] g_l^+(k) \,. \tag{4.27}$$

We wish to emphasize that the arbitrary constant C_2 has really been eliminated from the problem for all l values. It has been eliminated at high values of l by means of the convention $\delta_l(s_0) \xrightarrow[l\to\infty]{} 0$. The point here is that a knowledge of $\sin^2 \delta_l(s_1)$ is of course not sufficient to uniquely define δ_l. Once the choice of a convention is made for $\delta_l(s_1)$, the requirement of analyticity in l determines $\delta_1(s_1)$ from $\sin^2 \delta_l(s_1)$ for all l. Thus we see that the apparent arbitrariness introduced into the amplitude by the constant C_2 is completely removed by the assumption of analyticity in l.

5. – The resolvent Wiener-Hopf kernel.

We now proceed to construct an explicit expression for the resolvent kernel of the Wiener-Hopf eq. (3.3). This will enable us to establish the Fredholm character of eq. (3.9) and thus to obtain the complete solution for $N_l(s)$.

The following procedure for contructing the resolvent kernel was learned by the author from OMNÈS [13]. From eq. (4.26) we have that the Fourier transform, $g_l^+(k)$, of the solution, $N_l(x)$, is given by

$$g_l^+(k) = \frac{1}{\tilde{\varphi}_l^+(k)} \int\limits_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathrm{d}k' \frac{g_l^0(k')\,\tilde{\varphi}_l^-(k')}{2\pi i(k'-k)} \,, \tag{5.1}$$

where

$$\tilde{\varphi}_l^-(k) = \frac{\varphi_l^-(k)}{ik + (1-\alpha_l)} \,, \qquad \tilde{\varphi}_l^+(k) = \varphi_l^+(k)(-ik-\alpha_l) \,. \tag{5.2}$$

The solution, $N_l(x)$, is given by (4.27) and using the convolution integral [14],

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[13] R. OMNÈS: private communication.

[14] We are using here the formula

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int\limits_{-\infty}^{+\infty} f(y)\, h(x-y)\, \mathrm{d}y = \frac{1}{(2\pi)^{\frac{1}{2}}} \int\limits_{-\infty}^{+\infty} F(k) H(k) \exp[-ikx]\, \mathrm{d}k$$

where F and H are Fourier transforms of f and h; for details see P. M. MORSE and H. FESHBACH: *Methods of Theoretical Physics*, Part I (New York, 1953), p. 464.

we may infer that

$$n_l(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \mathrm{d}y\,\alpha^+(x-y)F^+(y)\,, \tag{5.3}$$

where

$$\alpha^+(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty+i(a_l+\varepsilon)}^{\infty+i(a_l+\varepsilon)} \frac{\mathrm{d}k}{\tilde{\varphi}_l^+(k)} \exp[-ik\xi] \tag{5.4}$$

and

$$F^+(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \mathrm{d}k\,\tilde{\eta}(k) \exp[-ik\xi] \tag{5.5}$$

with

$$\tilde{\eta}(k) = \frac{1}{2\pi i} \int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{\mathrm{d}k'\,g_l^0(k')\,\tilde{\varphi}_l^-(k')}{k'-k}\,. \tag{5.6}$$

We have used the « + » superscript on α and F to emphasize that they are generally nonzero only when their arguments are greater than or equal to zero. This is true because the Fourier transforms are holomorphic in the upper part of the k plane.

Using definitions (5.2) together with (4.7) it is easy to verify that

$$\tilde{\varphi}_l^{\pm}(k) \xrightarrow[k\to\infty]{} 1\,, \tag{5.7}$$

where the limit is taken in the domain of holomorphy. This means that α^+ defined through (5.4) has a δ function contribution which we separate out explicitly:

$$\alpha^+(\xi) = (2\pi)^{\frac{1}{2}}\delta(\xi) + \tilde{\alpha}^+(\xi)\,. \tag{5.8}$$

The function $\tilde{\alpha}^+(\xi)$ obviously has the Fourier transform

$$\frac{1}{\tilde{\varphi}_l^+(k)} - 1\,.$$

Using the definition of F^+ contained in eqs. (5.5) and (5.6), we may again call upon properties of the convolution integral [14] that will immediately enable us to deduce

$$F^+(\xi) = \frac{\theta(\xi)}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\beta\,\alpha^-(\xi-\beta)\,n_l^0(\beta)\,, \tag{5.9}$$

where $\theta(\xi)$ has the usual properties

$$\theta(\xi) = 0\,, \qquad \xi < 0\,,$$

$$\theta(\xi) = 1\,, \qquad \xi \geqslant 0\,,$$

and

(5.10) $$\alpha^-(z) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int \mathrm{d}k\, \tilde{\varphi}_l^-(k) \exp[-ikz]\,.$$

Again, as a consequence of (5.7) $\alpha^-(z)$ will have a δ-function part so we write in analogy with (5.8):

(5.11) $$\alpha^-(z) = (2\pi)^{\frac{1}{2}}\delta(z) + \tilde{\alpha}^-(z)\,.$$

As a consequence of eqs. (5.3) and (5.9) (using also (5.8) and (5.11)) we can write the following expression for the resolvent kernel, $R_l(x, x')$:

(5.12) $$R_l(x, x') = \theta(x')\delta(x - x') + \frac{\theta(x)\theta(x')}{(2\pi)^{\frac{1}{2}}}\tilde{\alpha}^-(x - x') + $$
$$+ \frac{\theta(x)\theta(x')}{(2\pi)^{\frac{1}{2}}}\tilde{\alpha}^+(x - x') + \frac{\theta(x)\theta(x')}{2\pi}\int_{\mathrm{Max}[0,x-x']}^{x} \mathrm{d}\xi\, \tilde{\alpha}^+(\xi)\,\tilde{\alpha}^-(x - \xi - x')\,,$$

where

(5.13) $$n_l(x) = \int_{-\infty}^{\infty} \mathrm{d}x' R_l(x, x') n_l^0(x')\,.$$

Fortunately, it is possible to make an explicit evaluation of the functions $\tilde{\alpha}_l^\pm$ in terms of the hypergeometric functions F. A straightforward calculation gives

(5.14) $$\tilde{\alpha}^+(z) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin^2 \pi a_l}{\sin^2 2\pi a_l}\Bigg[\exp[-a_l z]\frac{\Gamma^2(1 + a_l)}{\Gamma(1 + 2a_l)}\cdot$$
$$\cdot F(1 + a_l, 1 + a_l, 1 + 2a_l; \exp[-z]) -$$
$$- \exp[a_l z]\cdot\frac{\Gamma^2(1 - a_l)}{\Gamma(1 - 2a_l)} F(1 - a_l, 1 - a_l, 1 - 2a_l, \exp[-z])\Bigg]$$

and

(5.15) $$\tilde{\alpha}^-(z) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{\sin^2 \pi a_l}{\sin 2\pi a_l}\Bigg[\exp[(a_l + 1)z]\frac{\Gamma^2(1 + a_l)}{\Gamma(1 + 2a_l)}\cdot$$
$$\cdot F(1 + a_l, 1 + a_l, 1 + 2a_l; \exp[z]) -$$
$$- \exp[(-a_l + 1)z]\cdot\frac{\Gamma^2(1 - a_l)}{\Gamma(1 - 2a_l)} F(1 - a_l, 1 - a_l, 1 - 2a_l, \exp[z])\Bigg]\,.$$

Formula (5.12) may be compared with another formula for the resolvent kernel, which was recently derived by TEPLITZ (15) in a different manner from the above.

The presence of the δ-function in the formula (5.12) for the resolvent kernel is easily understood as TEPLITZ (15) has shown. If $a_l = 0$, then $\tilde{\alpha}^{\pm}(z) = 0$ and $n_l(x) = n_l^0(x)$. The problem then reduces to one of solving integral eq. (3.4), which is easily seen to be a Fredholm equation (16).

We originally assumed that we were at large values of l so that $a_l = \delta_l(s_1)/\pi < \frac{1}{2}$. If we now analytically continue to smaller values of l, it is easy to see that the resolvent kernel $R_l(x, x')$ is well defined and analytic for $a_l > \frac{1}{2}$. When a value of l is reached in the continuation procedure such that $a_l = \frac{1}{2}$, it would appear that R_l is singular because of the vanishing of the factor $\sin 2\pi a_l$ in the denominator of the function $\tilde{\alpha}^{\pm}$. This singularity is spurious, however, and it is easy to check that the numerators for $\tilde{\alpha}^{\pm}$ also vanish at $a_l = \frac{1}{2}$ so that the resolvent kernel is analytic at the point $a_l = \frac{1}{2}$, as TEPLITZ has emphasized (15).

6. – The Fredholm equation.

Having found an explicit expression for the resolvent Wiener-Hopf kernel, we now demonstrate that eq. (3.9) for $N_l^0(s)$ can be converted to Fredholm type whenever $a_l < 1$ and, hence, it possesses a unique solution. An integral equation is Fredholm for our purposes if its kernel is square integrable (SCHMIDT). Once $N_l^0(s)$ is known, we determine $N_l(s)$ through eq. (3.8) and our problem is solved. If we express the resolvent kernel $R_l(x, x')$ (eq. (5.12)) in terms of the variables s and s' (eq. (3.11)) and denote the resulting kernel by $R_l(s, s')$, we can write (eq. 3.9) in the form

$$N_l^0(s) = B_l^P(s) + \int_{s_0}^{s_1} \mathrm{d}s' \left[K_l(s, s') + \int_{s_0}^{s_1} \mathrm{d}\xi \frac{K_l(s, \xi)\tilde{R}_l(\xi, s')}{s_1 - s'} N_l^0(s') \right], \tag{6.1}$$

where $\tilde{R}_l$ is just R_l with the δ-function omitted. The kernel for the integral eq. (6.1) is obviously just

$$K_l(s, s') + \int_{s_0}^{s_1} \mathrm{d}\xi \frac{K_l(s, \xi)\tilde{R}_l(\xi, s')}{s_1 - s'} . \tag{6.2}$$

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(15) V. L. TEPLITZ: *Phys. Rev.*, **137**, B 136 (1965).

(16) In the Balazs model for the strip equations [ref. (3)] this is just what happens: $a_l = 0$ and the resulting equation for N is Fredholm.

The function $K_l(s, s')$ is square integrable since

$$(6.3)\qquad K_l(s, s') \xrightarrow[s,s'\to s_1]{} \text{const}\cdot\frac{(s_1-s')\ln(s_1-s')-(s_1-s)\ln(s_1-s)}{s'-s}\,.$$

To study the properties of the integral term in (6.2) we must establish the behavior of $R_l(\xi, s')$ as both arguments approach s_1. Utilizing eqs. (5.12), (5.14) and (5.15), it is readily concluded that

$$(6.4)\qquad \begin{cases} R_l(x, x') \xrightarrow[\substack{x\to\infty\\ x' \text{ fixed}}]{} \text{const}\cdot\exp[a_l x]\,,\\[2ex] R_l(x, x') \xrightarrow[\substack{x'\to\infty\\ x \text{ fixed}}]{} \text{const}\cdot\exp[(1-a_l)x']\,. \end{cases}$$

These limits are the same as those derived by CHEW [6], who assumed that $a_l<\frac{1}{2}$. The limits (6.3) continue to hold also if $a_l<1$. We established in the last Section that the resolvent kernel could be continued to the region $a_l>\frac{1}{2}$. Transforming to the (s, s') variables we find

$$(6.5)\qquad \begin{cases} \dfrac{\widetilde{R}_l(s, s')}{s_1-s} \xrightarrow[\substack{s\to s_1\\ s' \text{ fixed}}]{} \dfrac{\text{const}}{(s_1-s)^{a_l}}\,,\\[3ex] \dfrac{\widetilde{R}_l(s, s')}{s_1-s'} \xrightarrow[\substack{s'\to s_1\\ s \text{ fixed}}]{} \dfrac{\text{const}}{(s_1-s')^{a_l}}\,. \end{cases}$$

In Table I we give the limiting form of the kernel $\widetilde{R}(s, s')/(s_1-s')$ as both s and s' approach s_1 with $s>s'$, $s=s'$, $s<s'$.

TABLE I. – *Behavior of* $\widetilde{R}_l(s, s')/(s_1-s')$ *as* s *and* $s'\to s_1$.

	$s'>s$	$s=s'$	$s<s'$
$a_l<\frac{1}{2}$	$\text{const}\cdot(s_1-s)^{a_l}\,(s_1-s')^{a_l-1}$	$\text{const}\cdot(s_1-s)^{-1}$	$\text{const}\cdot(s_1-s)^{a_l-1}(s_1-s')^{-a_l}$
$a_l>\frac{1}{2}$	$\text{const}\cdot(s_1-s)^{-a_l}(s_1-s')^{-a_l}$	$\text{const}\cdot(s_1-s)^{-2a_l}$	$\text{const}\cdot(s_1-s)^{-a_l}\,(s_1-s')^{-a_l}$

Since the sum of two Schmidt operators is also Schmidt, in order to verify that (6.1) is Fredholm it is sufficient to establish that the integral term in the square brackets of (6.1) is square integrable in s and s'. Since also the product of two Schmidt operators is Schmidt this last conclusion will follow immediately if $\widetilde{R}_l(s, s')/(s_1-s')$ is Schmidt. Unfortunately Table I shows that this is not the case; $\tilde{R}_l(s, s')/(s_1 - s')$ is not square integrable. However if $a_l < \frac{1}{2}$ we can still show that the integral

$$\int_{s_0}^{s_1} \mathrm{d}\xi\, K_l(s, \xi)\, \tilde{R}_l(\xi, s')(s_1 - s')^{-1}$$

is square integrable. To do this we simply note from eq. (5.3) that $K_l(s, \xi)$ can be bounded as follows

$$K_l(s, \xi) \underset{s,s' \to s_1}{\leqslant} \mathrm{const} \cdot \ln(s_1 - s) \ln(s_l - \xi)\,. \tag{6.6}$$

So we find

$$\int_{s_0}^{s_1} \mathrm{d}\xi \frac{K_l(s, \xi)\tilde{R}_l(\xi, s')}{s_1 - s'} \leqslant \mathrm{const} \cdot \ln(s_1 - s)\left[(s_1 - s')^{-a_l}\int_{s_0}^{s'} \mathrm{d}\xi \ln(s_1 - \xi) \cdot (s_1 - \xi)^{a_l - 1} + (s_1 - s')^{1 - a_l}\int_{s'}^{s_1} \mathrm{d}\xi \ln(s_1 - \xi)(s_1 - \xi)^{-a_l}\right] \leqslant \mathrm{const} \cdot \ln(s_1 - s)(s_1 - s')^{-a_l} \tag{6.7}$$

which is, indeed, square integrable. Thus we can conclude that if $a_l < \frac{1}{2}$, eq. (6.1) is Fredholm. This result was given by CHEW [6].

But what if $a_l > \frac{1}{2}$? In this case our argument of the last paragraph will not go through. The kernel of the integral eq. (6.1) is not Fredholm as it stands. However the difficulty is easily repaired for we can consider the integral equation for the function $\bar{N}_l^0(s)$ defined by

$$\bar{N}_l^0(s) = N_l^0(s)(s_1 - s)^{-[a_l - \frac{1}{2} + \varepsilon]}, \qquad a_l < 1\,,\ 1 - a_l > \varepsilon > 0\,. \tag{6.8}$$

From eq. (6.1) we see that the integral equation that has for its solution $\bar{N}_l^0(s)$, has for its kernel

$$(s_1 - s)^{-[a_l - \frac{1}{2} + \varepsilon]} K_l(s, s')(s_1 - s')^{[a_l - \frac{1}{2} + \varepsilon]} + (s_1 - s)^{-[a_l - \frac{1}{2} + \varepsilon]} \cdot \int_{s_0}^{s_1} \mathrm{d}\xi\, K_l(s, \xi)\, \tilde{R}_l(\xi, s')(s_1 - s')^{a_l - \frac{3}{2} + \varepsilon}\,. \tag{6.9}$$

Utilizing the bound (6.6), the kernel (6.9) is easily seen to be square integrable.

With the results of this Section we may conclude that a solution to our original eq. (2.9) certainly exists whenever $a_l < 1$.

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7. – CDD ambiguities and maximal analyticity of the second degree.

Having shown in the last Section that solutions to the integral equations for the partial amplitude exist, at least for large l, we now turn to the questions of whether the solutions found are unique and whether unique solutions may be found for lower values of angular momentum using maximal analyticity of the second degree [10].

Let us review the logic of our current position. The basic mathematical problem to be solved is represented by eqs. (2.1) and (2.3):

$$B_l(s) = \frac{\sin\delta_l(s)\exp[i\delta_l(s)]}{\varrho_l(s)}, \tag{2.1}$$

$$B_l(s) = B_l^P(s) + \frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\operatorname{Im} B_l(s')}{(s'-s)}. \tag{2.3}$$

To complete the specification of the problem, we insisted that $\delta_l(s)$ and $B_l^P(s)$ be real in the gap (s_0, s_1) and we considered $B_l^P(s)$ to be a given function; we then sought to solve for $B_l(s)$ in the interval (s_0, s_1). Our next step consisted of writing $B_l(s)$ as $N_l(s)/D_l(s)$ where $D_l(s)$ was required to carry the phase of the amplitude on the interval (s_0, s_1) and be real elsewhere. That any $B_l(s)$ satisfying (2.1) and (2.3) could be so written was a consequence of the Omnès formula [9] (2.5). Our procedure was to define the properties of $D_l(s)$ and then to define $N_l(s)$ by $N_l(s) = B_l(s) D_l(s)$. The function $D_l(s)$ is, of course, not completely specified by its phase on the interval (s_0, s_1), we further required that $D_l(s) \to 1$ as $s \to \infty$. This led to eq. (2.6) for $D_l(s)$.

With eq. (2.6) for $D_l(s)$, we deduced the integral eq. (2.9) for $N_l(s)$ which we have shown to be soluble at least for $a_l < 1$. However, we could have equally well written for $D_l(s)$

$$D_l(s) = 1 + \frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{D_l(s')}{s'-s} + \sum_{i=1}^{n} \frac{\gamma_i}{s-\beta_i}, \tag{2.6'}$$

where the numbers γ_i and β_i are arbitrarily chosen complex numbers except that they must be picked so that $D_l(s)$ is real outside the interval (s_0, s_1). The equation for $N_l(s)$ then becomes

$$N_l(s) = B_l^P(s) + \sum_{i=1}^{n} \frac{\gamma_i[B_l^P(s) - B_l^P(\beta_i) + B_l(\beta_i)]}{s-\beta_i} + \frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{B_l^P(s') - B_l^P(s)}{s'-s} \varrho_l(s') N_l(s') \,. \tag{2.9'}$$

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Now it is clear that this equation for $N_l(s)$ will also have a solution (at least if none of the β_i^2 is in the interval (s, s_1)) as long as a_i is taken less than one. It then follows that our original problem (eqs. (2.1) and (2.3)) has infinitely many solutions which can be characterized by the parameters γ_i and β_i. What we are discussing, of course, is just the well-known CDD ambiguity (4). The result for $D_l(s)$ as computed by eq. (2.10) is no longer given by (2.5) but must be now written

$$(2.5') \qquad D_l(s) = \frac{(s-s_0)^{n-m}\prod_{i=1}^{m}(s-\mu_i)}{\prod_{i=1}^{n}(s-\beta_i)}\exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1}\mathrm{d}s'\,\frac{\delta_l(s')}{s'-s}\right],$$

where the μ_i denote the positions of the m zeros of $D_l(s)$. It also follows that the solution will have the property

$$(7.1) \qquad \delta_l(s_0) = (m-n)\pi\,,$$

which, interestingly, would just correspond to Levinson's theorem (17) if $\delta(\infty)=0$ and we were talking about an elastic problem.

The question now is: can some principle be invoked that selects only one of the many possible answers as the true solution? We start by establishing the manner in which solutions are uniquely determined at large values of l.

For sufficiently large values of l, we may appeal to the results of FROISSART (18) to conclude that no arbitrariness is possible. To be explicit, for large l the amplitude $B_l(s)$ in the interval (s_0, s_1) must be determined in terms of the absorptive parts in the crossed t and u channels by the FROISSART-GRIBOV transform (19)

$$(7.2) \qquad B_l^{\pm}(s) = \frac{1}{\pi}\int_{t_0}^{\infty}\frac{\mathrm{d}\xi}{2\nu^{l+1}}\,Q_l(1+\xi/2\nu)\,D^{\pm}(\xi, s)\,,$$

where the $\pm$ refer to the signature of the amplitude in the usual way and $D^{\pm}(\xi, s)$ is a linear combination of the absorptive parts in the t and u channels.

It is clear that in our problem the input function $B_l^P(s)$ is determined by $D^{\pm}(\xi,s)$ for we have only to subtract the cut contribution (s_0, s_1) from (7.2) in order to

(17) N. LEVINSON: *Kgl. Danske Videnskab, Selskab, Matt.-Fys. Medd.*, **25** No. 9 (1949).

(18) M. FROISSART: *Phys. Rev.* **123**, 1053 (1961).

(19) M. FROISSART: *Report to the La Jolla Conferenc eon Theoretical Physics*, (June 1961) (unpublished); V. N. GRIBOV: *Žurn. Ėksp. Teor. Fiz.*, **41**, 667, 1962 (1961) [English transl.: *Sov. Phys. JETP*, **14**, 478, 1395 (1962)].

have $B_l^P(s)$. Now it certainly follows that our solution if it is to be the correct one and, hence, agree with (7.2), cannot depend arbitrarily on parameters like γ_i and β_i of eq. (2.6′). It could happen, of course, that the correct equation to solve was (2.9′) but, if so, the β_i and γ_i would not be arbitrary. They would have to be given in some way in terms of $D(\xi, s)$. We now proceed to show that the correct equation is, in fact, the original eq. (2.9) without poles in the D-function.

The point is that eq. (7.2) has the important characteristic that for $l \to \infty$ $B_l(s) \to 0$ for all s in (s_0, s_1). With the phase-shift convention we have consistently chosen throughout, we know that $\delta_l(s_1) \to 0$ as $l \to \infty$. But $\delta_l(s_0) = (m - n)\pi$ and in order to fulfill the requirement that $B_l(s)$ vanish in the limit of large l, we clearly must have $m = n$. But we also know that there are no bound states for large angular momentum so $n = 0$. Thus the correct solution for large l is the one for which D has no poles.

Now that we have found the correct and unique solution to our problem at large l, it is now important to establish the solutions for lower l values. This is now done by analytic continuation l and as a result of assuming maximal analyticity of the second degree (MASD). That is, we shall try to establish that our solution is analytic in l and hence define the partial-wave amplitudes by analytic continuation. It is the assumption of MASD that insists that the physical amplitudes agree with the continuations.

Our first consideration is the integral equation (6.1). It is important for our purposes to know that the solution $N_l^0(s)$ defines an analytic function of l. TIKTOPOULOS [20] has made a careful study of the analyticity of solutions to Fredholm equations whose kernels depend on a complex parameter l. A sufficient set of circumstances for the solution to be analytic in some domain M of the variable l are these: 1) the kernel be an analytic function of l in M; 2) the kernel be bounded over M by a square integrable function which is independent of l. The exact form of the kernel of eq. (6.1) is, of course, not known to us, but it is an analytic function of l with probably only isolated singularities. The closest singularity of this kernel known to exist on the real axis for s in the interval is a pole at $l = -1$. Otherwise, it is reasonable to assume that the kernel will be analytic in some neighborhood of the real axis for $\operatorname{Re} l > -1$ and s in the range (s_0, s_1). Having satisfied criterion 1) above it only remains to show the kernel of eq. (6.1) can be appropriately bounded. That this is true follows from an examination of Table I and eq. (6.6), which give the bounds of K_l and R_l near the dangerous point where both arguments approach s_1.

([20]) G. TIKTOPOULOS: *Phys. Rev.*, **133**, B 1231 (1964). The appendix of this paper summarizes the mathematical theorems which are important for our considerations. MANDELSTAM has considered the analyticity properties in l of N/D equations when elastic unitarity is assumed: S. MANDELSTAM: *Ann. Phys.*, **21**, 302 (1963).

Thus we have $N_l^0(s)$ as an analytic function of l at least for $\operatorname{Re} l$ such that $a_l < -1$ and $\operatorname{Re} l > 1$. The analytic function $N_l^0(s)$ is, in fact, just the unique solution of the Fredholm equation (6.1). Once we have $N_l^0(s)$ specified as an analytic function of l we define $N_l(s)$ as an analytic function of l through eq. (3.8):

$$N_l(s) = \int_{s_0}^{s_1} \mathrm{d}s'\, O_l(s, s')\, N_l^0(s') \,. \tag{3.8}$$

The kernel $O_l(s, s')$ is an analytic function and the transform (3.8) certainly defines $N_l(s)$ as an analytic function of l wherever the integral exists. This region certainly includes the domain $a_l < 1$. Finally, $D_l(s)$ is defined over the same domain by eq. (2.10):

$$D_l(s) = 1 - \frac{1}{\pi} \int_{s_0}^{s_1} \frac{\mathrm{d}s'}{s' - s}\, \varrho_l(s')\, N_l(s') \,. \tag{2.10}$$

Zeros of $D_l(s)$ corresponding to bound states are to be expected, in general, as l is decreased. It is, interesting to display the mechanism by which eq. (2.5) for $D_l(s)$ develops a zero ([21]). It must be a smooth operation if $D_l(s)$ is to be an analytic function of l. In particular, we wish to verify that $D_l(s)$ maintains its normalization to one as l is decreased even when zeros develop. What happens is that for each zero $(s - s_i^B(l))$ which develops in D there will simultaneously occur a factor of $(s - s_0)$ in the denominator.

The zeros of $D_l(s)$ which generally represent poles of the amplitude must emerge from the second sheet of $D_l(s)$. This happens as follows. If we continue down through the (s_0, s_1) cut of the amplitude we find a pole at $s = s^B(l)$ (we assume to begin with that D has no zero and we observe the process by which the first zero moves onto the physical sheet). In the neighborhood of this pole on the unphysical sheet we may write

$$\begin{cases} \exp[2i\delta_l(s)] \xrightarrow[s \to s^B]{} \dfrac{\Gamma(l)}{s - s^B(l)} \,, \\[2ex] \delta_l(s) \xrightarrow[s \to s^B]{} \dfrac{-1}{2i} \ln\left(s - s^B(l)\right) . \end{cases} \tag{7.3}$$

Thus $\delta_l(s)$ is logarithmically singular at $s^B(l)$ at a point reached by continuing down through the cut (s_0, s_1). (There is a similar singularity at the point $[s^B(l)]^*$ reached by continuing up from the bottom through the (s_0, s_1) cut but this singu-

([21]) For a detailed study of the motion of resonance poles with angular momentum, see C. E. JONES: *Ann. Phys.*, **31**, 481 (1965).

larity will not concern us here.) We now decrease l and shall assume that this causes the pole at $s^B(l)$ to move out onto the physical sheet.

Let us now examine $D_l(s)$ as defined by eq. (2.5) (we assume originally that D has no zeros):

$$D_l(s) = \exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\delta_l(s')}{s'-s}\right]. \tag{2.5}$$

We may regard the integral over δ_l as an integration over a contour C with fixed endpoints at $s'=s_0$ and $s'=s_1$. For physical values of s, the integral (2.5) is defined by giving s a small positive imaginary part. In order to evaluate $D_l(s)$ on the second sheet in the neighborhood of the points $s=s^B(l)$, we must distort the contour C as shown in Fig. 1 and fold it around the logarithmic branch point of $\delta_l(s')$ at $s'=s^B(l)$.

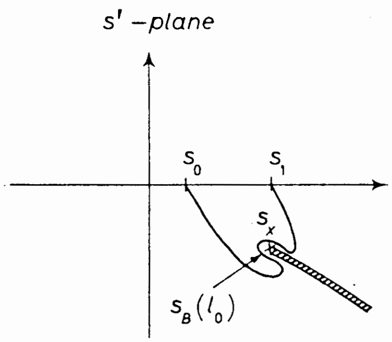

Fig. 1. – Evaluation of $D_l(s)$ near $s=s^B(l)$.

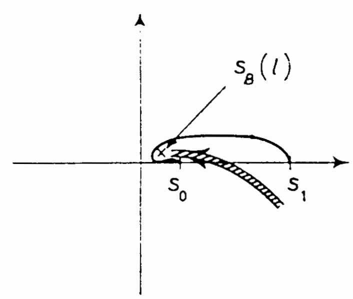

Fig. 2. – Emergence of a zero of $D_l(s)$ onto the physical sheet.

Near $s=s^B(l)$, the most important contribution to the integral for $D_l(s)$ comes from that part of the contour near the cut in Fig. 1. To evaluate this we only need to use the discontinuity of the logarithm (7.3). We thus obtain

$$D_l(s) \xrightarrow[s\to s^B(l)]{} \exp\left[-\frac{1}{\pi}\int_{s^B(l)} \frac{\mathrm{d}s'}{s'-s}\,\frac{2\pi i}{2i}\right] = \mathrm{const}\cdot\left(s-s^B(l)\right). \tag{7.4}$$

So we see explicitly that $D_l(s)$ has a zero on its second sheet at the location of a pole of the amplitude. As we decrease l in such a way that $s^B(l)$ moves onto the physical sheet the contour C becomes distorted as shown in Fig. 2.

The contour from $s^B(l)$ to s_0 can be easily evaluated and the result is

$$D_l(s) = \frac{s-s^B(l)}{s-s_0}\exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1} \mathrm{d}s' \frac{\delta_l(s')}{s'-s}\right].$$

If there are k bound states, we have

$$(2.5) \qquad D_l(s) = \prod_{i=1}^{k} \frac{s - s_i^B(l)}{(s - s_0)^k} \exp\left[-\frac{1}{\pi}\int_{s_0}^{s_1} ds' \frac{\delta_l(s')}{s' - s}\right],$$

where $s_i^B(l)$ are the location of the k bound states. We also see that $\delta_l(s_0) = k\pi$, again strongly reminiscent of Levinson's theorem ([17]) for an elastic problem where $\delta(\infty) = 0$.

Thus we see that as l is decreased, the zeros of $D_l(s)$ emerge in a smooth fashion onto the physical sheet and $D_l(s)$ maintains its normalization to one at infinity. Also the original eq. (2.9) for $N_l(s)$ continues to be valid even when there are bound state poles.

Of course, it is also clear from eq. (2.10) that $D_l(s)$ will maintain its normalization to one at least so long as the integral over $N_l(s)$ converges.

We have seen that $D_l(s)$ cannot have poles for sufficiently large angular momenta. We would now like to verify that as we decrease the angular momentum poles cannot emerge from the second sheet of $D_l(s)$ in a manner like the one just described for the zeros.

In order for $D_l(s)$ to have a pole on its second sheet, we clearly need only to change the sign of the logarithmic singularity of the phase shift in eq. (7.3). But if the phase shift has the behavior

$$\delta_l(s) \xrightarrow[s \to s^P]{} \frac{+1}{2i} \ln (s - s^P(l)) ,$$

where s^P is on the second sheet, reached by going through the (s_0, s_1) from above, then this means the S matrix $\exp[2i\delta_l(s)]$ vanishes on the second sheet at $s = s_p$. This in turn means that the S-matrix and hence the amplitude have a *pole* on the *physical* sheet at $s = s_p$. It may be possible to dream up a B_l^P which produces such a situation but we reject such a possibility as physically unrealistic since it produces poles on the physical sheet and also violates the condition that the Regge pole $\alpha(s)$ has the property $\operatorname{Im} \alpha(s) \geq 0$ for $s > s_0$. We may, therefore, safely conclude that the D-function will not develop poles as the angular momentum is decreased. Also from another point of view, we can understand that if $N_l(s)$ is a solution of (3.3) then the integral in eq. (2.10) over $N_l(s)$ to determine $D_l(s)$ must converge and no poles of $D_l(s)$ can emerge from the cut (s_0, s_1).

This leads us naturally to the question of what happens if, as l is decreased, we encounter a region where $a_l > 1$. If this is the case then the eq. (6.1) for $N_l(s)$ ceases to be Fredholm and nearly all our questions for $N_l(s)$ and $D_l(s)$ fail to be defined. Of course, in the domain where $a_l < 1$ we have established that $N_l(s)$ and $D_l(s)$ will be analytic functions of l and, in fact, we have given a prescription for uniquely calculating $N_l(s)$ and $D_l(s)$ in terms of the input function $B_l^P(s)$ using a combination of Fredholm and Wiener-Hopf theory. We may say that we now simply analytically continue these solutions for $a_l < 1$ into the region $a_l > 1$. While this statement may be all right in principle, it is somewhat abstract and we would prefer to have explicit formulas giving the analytic continuation and showing how to calculate $N_l(s)$ and $D_l(s)$ at values of l with $a_l > 1$. In seems certain that such formulas can be derived using the techniques of TIKTOPOULOS [20]. We shall not, however, give such formulas here, since there appears no need in practice to explicitly compute the amplitude in these regions.

We shall, however, ask another question which *is* of practical importance and which is closely related. Suppose that as we continue to decrease l a_l once again drops below one, can we conclude that all our equations which now once again become well-defined are correct? Can we once again calculate the amplitude using the techniques devised herein. The answer is yes. The amplitude continues to be unitary and, hence, the basic N and D equations are correct. The only real matter we must decide is again whether poles should be added to $D_l(s)$ giving for the N eq. (2.6′). However, we have previously argued that $D_l(s)$ cannot acquire poles as a result of analytic continuation in l and this argument did not depend upon a_l being less than one. So we conclude that the solutions to our problem whenever $a_l < 1$ are correctly given by the solutions to the integral equations as we have formulated them without poles in $D_l(s)$.

8. – Concluding remarks.

We have shown how the principle of analytic continuation in l or maximal analyticity of the second degree [10] may be used to remove arbitrariness of the CDD type from partial-wave amplitudes. Using N/D equations with a finite strip for the D-function has made it possible to carry out a discussion without assuming elastic unitarity in inelastic regions. It may also be added that any other types of arbitrariness such as, for example, the s-wave subtraction constant have also been eliminated since the amplitudes are completely determined by the input $B_l^P(s)$.

We now wish to bring attention to a point which has not been emphasized thus far. If we assume the function $B_l^P(s)$ to be given exactly, then our problem of finding $B_l(s)$ is actually already solved by the following simple process: compute $\text{Im}\, B_l^P(s)$ for $s > s_1$, which is just the discontinuity of $B_l(s)$ for $s > s_1$;

analytically continue the discontinuity $\text{Im}\, B_l^P(s)$ to the region $s < s_1$. Since s_1 was assumed to be less than the first inelastic threshold, we may determine $B_l(s)$ in the region $s_0 \leqslant s \leqslant s_1$ and so determine $B_l(s)$ through formula (2.3).

What, then, was the point of our whole discussion? First of all, the fact that our complicated integral equations for $B_l(s)$ have unique solutions does not follow from the above. The existence of the solutions to the integral equations was not based upon $B_l^P(s)$ or its discontinuity being analytic in s. Thus, although we could be sure that the «correct» answer calculated as outlined above would be a solution to our integral equations, we had no assurance that it would be the only solution. In fact, the first type of arbitrariness in the solutions to our integral equation is relevant here. We saw in Sect. **4** that there existed an infinite number of solutions to our Wiener-Hopf equation corresponding to the addition of multiples of the homogeneous solution, Although as we discussed all but one of these solutions had the wrong analyticity properties in l, these extra solutions also differed from the «correct» answer by having a singularity at $s = s_1$. These extra solutions are thus obviously different than would be calculated from the solution one obtains by the analytic continuation of the discontinuity in energy.

Another deeper point has to do with the assumption of the existence of solutions. No one has, in fact, shown that there exist functions satisfying all the requirements of analyticity and unitarity which we impose. Most discussions assume such functions exist and, under this assumption, display relations or equations that such functions must obey. The N/D equations embody the enforcement of unitarity utilizing also some analyticity requirements. It is a fortunate state of affairs that, under these conditions, the existence of a solution to the integral equations can be established without a detailed knowledge of the input function $B_l^P(f)$. If such a solution could not be found, our original assumption of the existence of solutions to the exact problem might fall into question.

Finally, there is the immensely practical point that we do not, in fact, know the function $B_l^P(s)$ exactly in an actual calculation. But if the approximate $B_l^P(s)$ possesses reasonable analyticity in l and s all the requirements for unique solutions, analytic in l, will be fulfilled and the approximate problem will parallel the exact situation in a useful and important way. It is obvious that in the approximate problem we could not achieve a unitary solution by continuing an approximate discontinuity $\text{Im}\, B_l^P(s)$ as discussed above.

We may just add the remark that in the approximate practical calculation outlined by CHEW [1] and CHEW and JONES [2] all requirements are met for determining a unique solution. In fact in their model s_1 is chosen so that a is always less than unity so the results of this paper are immediately applicable As mentioned before the Balazs model involves setting $a_l = 0$.

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RIASSUNTO (*)

Si espone qui un trattamento quasi completo delle equazioni N/D in cui la funzione D ha solo un taglio finito. Si passa in rassegna il lavoro originale di Chew su queste equazioni e lo si sviluppa per provare l'esistenza di soluzioni ogni qualvolta $\delta_l(s_1) < \pi$ e per ridurre in questi casi l'equazione integrale per N ad un tipo combinato di Wiener-Hopf-Fredholm. Si ottiene una formula esplicita per il nocciolo risolvente di Wiener-Hopf che comporta un solo integrale su prodotti di funzioni ipergeometriche. Si esaminano le ambiguità CDD e si dimostra l'analiticità massima del secondo grado come un mezzo per eliminare ogni ambiguità dalla soluzione.

(*) *Traduzione a cura della Redazione.*